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FREQUENCY INDEPENDENT ANTENNAS

CONFERENCE ROOM

by

V. H. RUMSEY

25 October 1957

Contract No. AF33(616)-3220
Project No. 6(7-4600) Task 40572
WRIGHT AIR DEVELOPMENT CENTER



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ABSTRACT

There is a class of antennas whose pattern as well as impedance is practically independent of frequency for all frequencies above a certain value. The general formula for their shape is $r = e^{a(\varphi + \varphi_0)} F(\theta)$ where r, θ, φ are the usual spherical coordinates, a and φ_0 are constants and $F(\theta)$ is any function of θ . Assuming a to be positive, φ ranges from $-\infty$ to some finite value which determines the low frequency limit. For such antennas, a change of frequency is equivalent to a rotation of the antenna about $\theta = 0$. It appears that the pattern converges to the characteristic pattern as the frequency is raised, if a is not infinite, and that the impedance converges to the characteristic impedance for all a .

ACKNOWLEDGEMENT

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1. ANTENNAS SPECIFIED BY ANGLES

To introduce the idea of frequency independent antennas, let us consider the familiar problem of modelling an antenna to a different wavelength. As we know, if all the dimensions of a lossless antenna are increased by some factor K , the pattern and impedance remain fixed if the operating wavelength is also increased by the factor K . In other words, the performance of the antenna is independent of frequency if its dimensions measured in wavelengths are held constant. It follows that if the shape of the antenna were such that it could be specified entirely by angles, its performance would be independent of frequency. The infinite biconical antenna¹ is the most familiar example: it is specified by the angles of the two cones and the angle between their axes. However, there is an infinite variety of shapes which are completely specified by angles, and these form the starting point for the design of frequency independent antennas. They must all extend to infinity (because if they did not they would have at least one characteristic length), and therefore they do not immediately lead to practical designs. The key problem is therefore to determine how rapidly, if at all, the performance of the finite structure converges to that of the infinite structure. Let us, however, defer consideration of this question until we have explored some applications of the "angle method" to familiar antennas,

The problem of a typical directional antenna can be illustrated by considering a unipole in front of a plane reflector as shown in Fig. 1: it is specified by the lengths $ABCD$ and ℓ . One of the major limitations on its pattern and impedance bandwidth is represented by the distance D from the unipole to the reflector. By changing the reflector and unipole to coaxial cones, all dimensions except B and ℓ are replaced by angles. This simple application of the angle method does indeed give significant improvement of the impedance and pattern bandwidth.² The impedance is practically constant above a certain frequency, but unfortunately the pattern is not. This appears to be typical of most conventional antennas designed according to the angle method. However, there is a whole class of unconventional antennas whose pattern, as well as impedance, remains

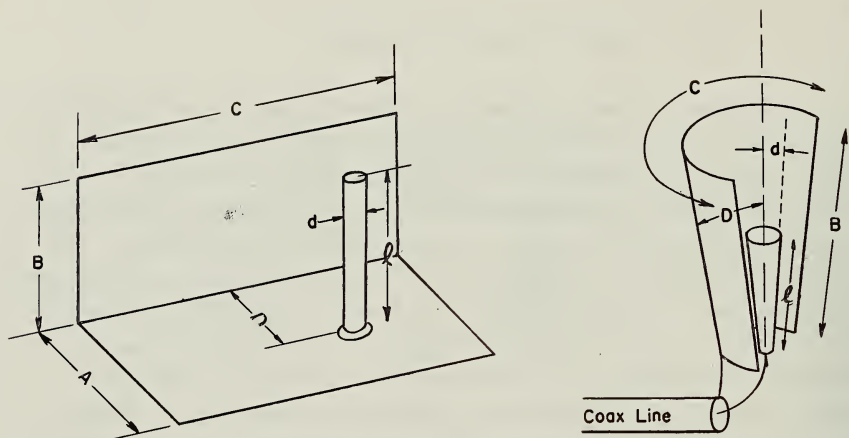


FIGURE 1

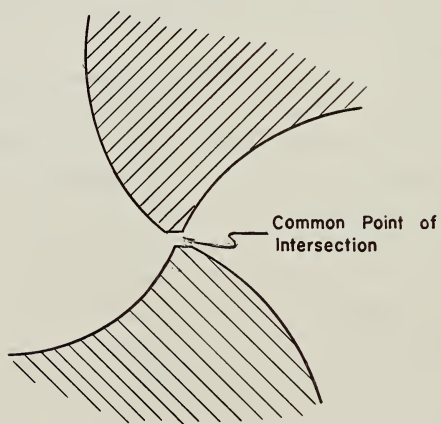


FIGURE 2

practically constant above a certain frequency. In view of the fact that all experience suggests that the pattern of any antenna develops more and sharper lobes as the frequency is increased, this result is indeed remarkable.

2. THE GENERAL APPROACH

Let us consider, then, the entire class of antenna shapes which are specified by angles. To begin with, let us restrict the discussion to plane sheet antennas; that is, we consider all plane curves which remain essentially the same when scaled to a different unit of length. Such curves can be used to determine the shape of a plane sheet antenna, by taking the input terminals at the common point of intersection of four curves, as illustrated in Fig. 2. Obviously we do not want to have to move the terminals when we change frequency, so we add the restriction that the location of the terminals remain fixed. The fact that a typical curve remains essentially unchanged by a change of scale implies that the new curve can be made to coincide with the old one by translation and rotation. Since a translation is eliminated by the requirement that the terminals stay fixed, the problem is to determine all curves such that a change of scale is equivalent to a rotation. This can be stated symbolically in the form

$$Kr(\varphi) = r(\varphi + C) \quad (1)$$

where $r(\varphi)$ denotes the radius r as a function of the polar angle φ , K is the scale change and C the angle of rotation to which it is equivalent. Thus K depends on C , but K and C are independent of φ (and r).

$$\therefore r(\varphi) \frac{dK}{dC} = \frac{\partial r(\varphi + C)}{\partial C} \quad (2)$$

$$\text{and} \quad K \frac{dr(\varphi)}{d\varphi} = \frac{\partial r(\varphi + C)}{\partial \varphi} \quad (3)$$

$$\text{But} \quad \frac{\partial r(\varphi + C)}{\partial C} = \frac{dr(\varphi + C)}{d(\varphi + C)} = \frac{\partial r(\varphi + C)}{\partial \varphi} \quad (4)$$

$$\therefore r(\varphi) \frac{dK}{dC} = K \frac{dr(\varphi)}{d\varphi} \quad (5)$$

$$\text{or} \quad \frac{dr}{d\varphi} = ar \quad (6)$$

where a is independent of φ :

$$a = \frac{1}{K} \frac{dK}{dC} . \quad (7)$$

It follows from (6) that

$$r = r_0 e^{a\varphi} \text{ where } r_0 \text{ is a constant.} \quad (8)$$

$$\text{Let } r_0 = e^{a\varphi_0} \text{ where } \varphi_0 \text{ is a constant} \quad (9)$$

$$\text{or } a\varphi_0 = \ln r_0 . \quad (10)$$

$$\text{Then } r = e^{a(\varphi + \varphi_0)} \quad (11)$$

$$\text{or } \varphi + \varphi_0 = \ln r^{1/a} . \quad (12)$$

We recognize (11) or (12) as the formula for an equiangular spiral: it contains the two parameters, a , which represents the rate of expansion, and φ_0 , which represents the orientation. Thus the shape of all plane sheet frequency independent antennas must be defined by equiangular spirals. (Note that when $\frac{1}{a} = 0$ the spiral degenerates into the straight line $\varphi = -\varphi_0$.) Theoretically it might appear that we could obtain four curves such as those shown in Fig. 2 by selecting four different combinations of a and φ_0 , but it is easily verified that unless a is the same for all, such curves overlap at infinitesimal values of r , thus placing a short circuit across the terminals. It is therefore necessary to choose four different φ_0 with the same a .

The general problem is to find all surfaces which have the property that a change in the unit of length is equivalent to a certain rotation. Then if we construct a metal antenna whose surface is one of these surfaces, its performance will be the same at all wavelengths except for a rotation of the coordinate system. This problem is analyzed in the appendix. The solution can be written in the form

$$r = e^{a(\varphi + \varphi_0)} F(\theta) \quad (13)$$

where in principle $F(\theta)$ can be any function of θ . The shapes represented by (13) can be very complicated because in general an increase of 2π in φ does not give the same r : as φ ranges from $-\infty$ to ∞ the surface weaves around through all space. Figure 3 illustrates a simple example which

gives a practical antenna design. Figure 4 illustrates the case where $\ln F(\theta)$ is periodic in θ with period $2\pi a$. This gives a simple surface like a screw thread which is uniformly expanded in proportion to the distance from the origin: an increase of 2π in φ is equivalent to moving one turn along the screw. The plane antennas considered in the paper by R.H. DuHamel and D.E. Isbell³ represent a cross section through the axis of this kind of antenna.

3. THE PATTERN

To examine the question of pattern convergence, we make use of a characteristic of the field which depends on the fact that a rotation is equivalent to an expansion. Since we are dealing with the frequency independent mode, the analysis of the field can be simplified by considering the static or DC case. For this case it has been shown by P.E. Mast⁴ that for the plane antenna the field is a function of the two variables $S = re^{-a\varphi}$ and θ , rather than the three variables r, φ, θ . (Note that S is constant on any equiangular spiral in the family characterized by the parameter a). This result implies that the direction of the current on the antenna is along the lines $S = \text{constant}$; i.e., the currents follow a spiral flow. The simplest way to analyze the pattern is to make use of the intuitive idea of coupling between adjacent turns of this spiral flow. Assuming, to begin with, that the AC current distribution is like the DC distribution traveling with the free space phase velocity, we see that a "resonance" will build up in the current distribution where the mean circumference of the spiral is about one wavelength. This resonance will be highly damped by radiation so that most of the current is dissipated in the first resonance. The pattern due to the first resonance is therefore almost the same as the pattern due to the infinite structure. Admittedly this argument is weak, but it is not altogether worthless because it does give a simple way of estimating the pattern which agrees satisfactorily with measurements on finite antennas. It shows that the condition for pattern convergence is that the current flow be spiral rather than rectilinear, i.e., that the parameter a should not be infinite.

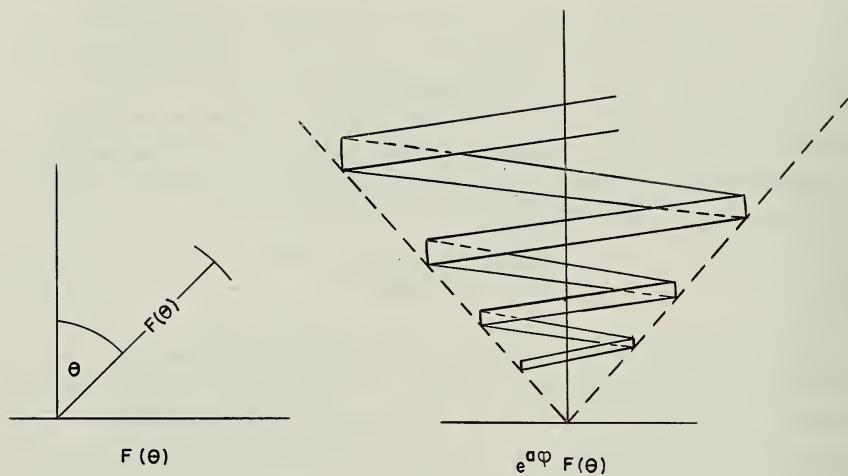


FIGURE 3

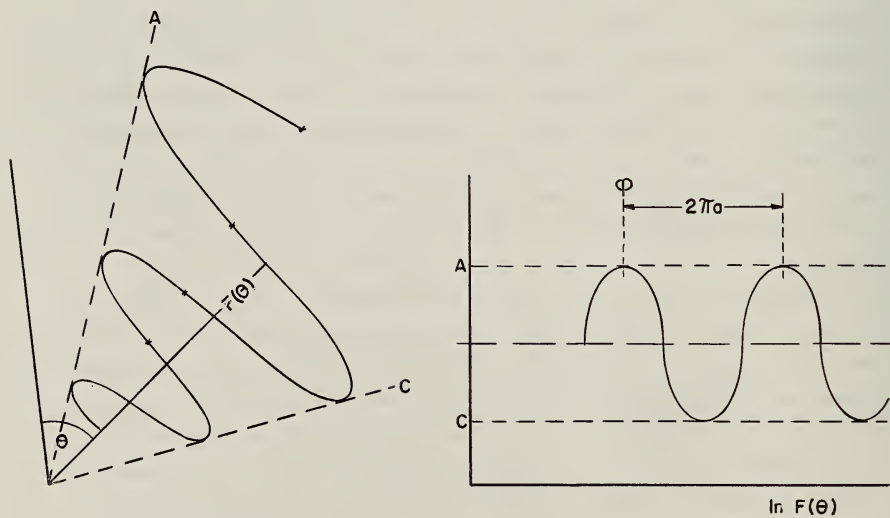


FIGURE 4

These theoretical conjectures are borne out in practice. The case $a = \infty$ represents the biconical type of antenna for which the pattern does not converge. Various cases for $a \neq \infty$ have been investigated by J.D. Dyson⁴ and R.L. Carrel,⁵ and convincing evidence of the constancy of the pattern over a 20:1 frequency range has been obtained. Note that a change of frequency is equivalent to a rotation. The pattern at one frequency, f_1 , is the same as the pattern at another frequency, f_2 , if the coordinate system is rotated about the $\theta = 0$ axis through an angle $\frac{1}{a} \ln \frac{f_1}{f_2}$: the pattern scans around the $\theta = 0$ axis at a rate which depends on a . If measurements are made in a fixed plane, the pattern in general varies with frequency, and the frequency independent nature of the complete spherical pattern may be missed in such measurements.

4. THE IMPEDANCE

The convergence of the input impedance to a constant value as the frequency is increased is familiar in connection with the biconical antenna. It has been confirmed in all cases of the general type represented by (13) which have been investigated.^{4,5} The determination of the characteristic impedance is an important problem in connection with frequency independent antennas. Schelkunoff's well-known formula for the characteristic impedance of two coaxial cones¹ is

$$377 \ln \left(\tan \frac{A}{2} \cot \frac{B}{2} \right)$$

where A and B are the cone angles measured from the common axis. The formula for two inclined cones is⁶

$$377 \left\{ \tanh^{-1} \left[\frac{\tan \frac{\varphi_1}{2}}{\tan \frac{\psi}{2}} \right] + \tanh^{-1} \left[\frac{\tan \frac{\varphi_2}{2}}{\tan \frac{\psi}{2}} \right] \right\}$$

where φ_1 , φ_2 and ψ are defined in Fig. 5. The formula for a symmetrical plane sheet antenna consisting of two triangles with a common apex at the terminals⁷ is

$$189 \frac{K(\cos \psi)}{K(\sin \psi)}$$

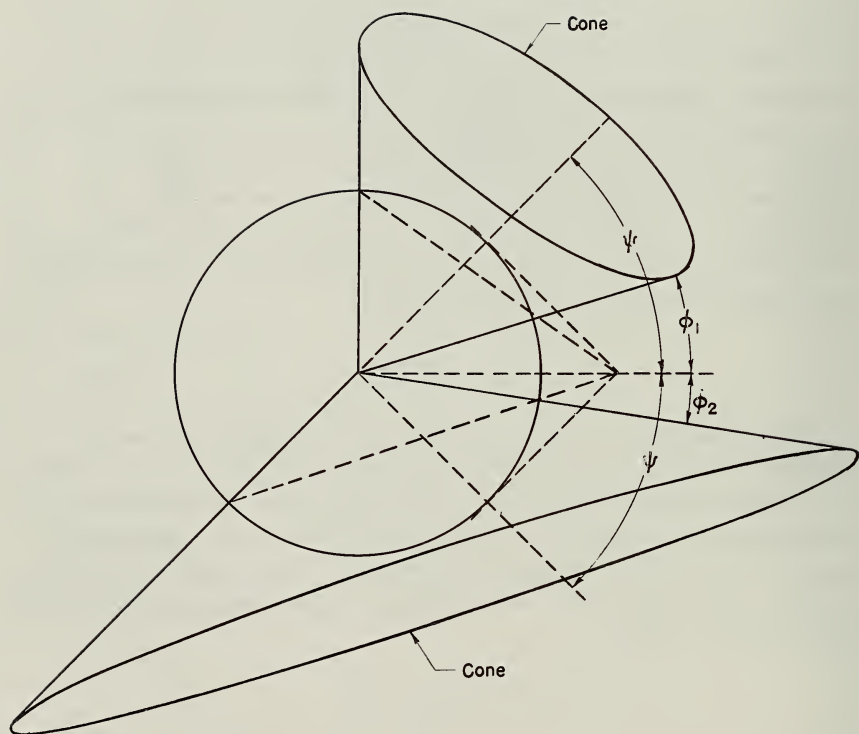


FIGURE 5

where K represents the elliptic integral defined by

$$K(x) = \int_0^1 \frac{dt}{(1-t^2)(1-x^2t^2)}$$

and ψ represents the half angle of the triangular strips measured from the common axis.

In connection with the impedance, we should note an interesting property which was pointed out by Mushiake in one of the Tohoku University reports. It is that the impedance of any plane sheet antenna whose shape is the same as the shape of its complement (except for a trivial change of coordinates) is independent of frequency and equal to $60\pi = 189$ ohms. The complementary antenna is defined as the portion of a metal plane which is not covered by the original antenna: when the antenna and its complement are fitted together they completely cover the whole plane without overlapping. The constant impedance of a "self-complementary" antenna follows from the relation⁸

$$Z_1 Z_2 = (60\pi)^2$$

between the impedance Z_1 of the antenna and the impedance Z_2 of its complement. Figure 6 gives some examples of self-complementary shapes.

5. PSEUDO-FREQUENCY-INDEPENDENT ANTENNAS

The idea of a pseudo-frequency-independent antenna is illustrated by the horn antenna shown in Fig. 7. It consists of metal sheet perforated by holes of uniformly expanding size: any hole is exactly like its neighbor on the left except for a fixed expansion. The idea is that the effective size of the horn remains roughly independent of wavelength because the metal sheet becomes approximately transparent once the holes become greater than about half a wavelength square. More precisely, it can be seen that if the horn started from a point and extended to infinity, it would "look" the same to any two wavelengths whose ratio was equal to the expansion factor. Some interesting examples of pseudo-frequency-independent antennas are described in the paper³ by R. H. DuHamel and D. E. Isbell.

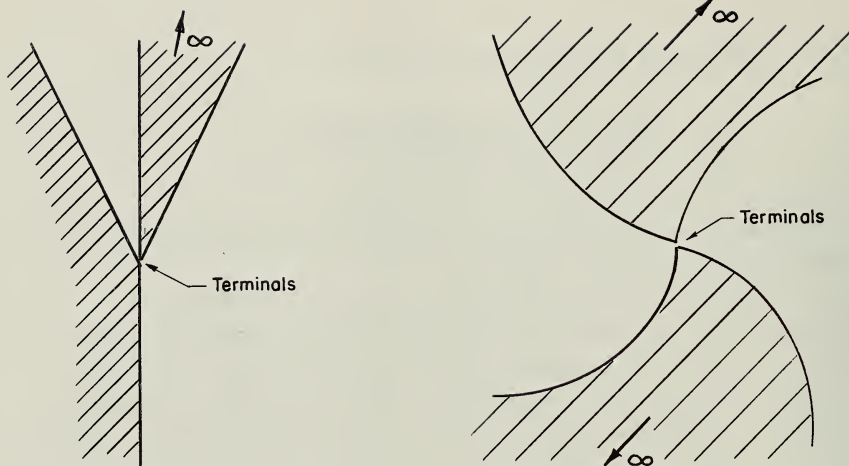


FIGURE 6

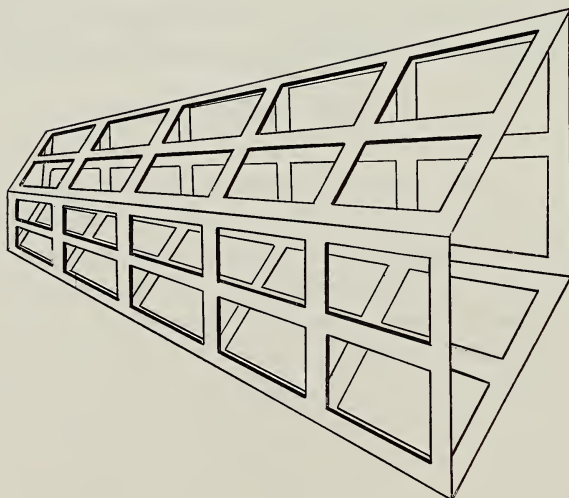


FIGURE 7

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APPENDIX

SURFACES FOR WHICH A ROTATION IS EQUIVALENT TO AN EXPANSION

A surface can be represented by the equation

$$r = f(\theta\varphi) \quad (14)$$

where r , θ and φ are the conventional spherical coordinates. The condition that a uniform expansion be equivalent to a rotation can be written as

$$K f(\theta\varphi) = f(\theta'\varphi') \text{ or } Kr = r' \quad (15)$$

where K is the expansion factor and $(\theta'\varphi')$ represents the direction after rotation of the line represented by $(\theta\varphi)$, e.g., for a rotation about the z axis ($\theta = 0$), $\theta' = \theta$, and $\varphi' = \varphi + c$. For a general rotation, we express the connection in the following way, in order to have the advantage of working with linear transformations. We define

$$\theta_1 = \theta; \theta_2 = \varphi \quad (16)$$

$$\rho_1 = \sin \theta_1 \cos \theta_2; \rho_2 = \sin \theta_1 \sin \theta_2; \rho_3 = \cos \theta_1 \quad (17)$$

$$\underline{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \quad (18)$$

Thus $\underline{\rho}$ represents the rectangular coordinates of the point (θ_1, θ_2) on the unit sphere: $\underline{\rho}'$ is similarly defined with respect to θ_1', θ_2' .

Then the connection between θ_1, θ_2 and θ_1', θ_2' can be expressed as

$$\underline{\rho}' = T \underline{\rho} \quad (19)$$

where T is a (3×3) matrix independent of $\theta_1, \theta_2, \theta_1', \theta_2'$. To express T we introduce

$$J(c) = \begin{pmatrix} \cos c & \sin c & 0 \\ -\sin c & \cos c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (20)$$

It is easily verified that $\mathcal{J}(c) \underline{\rho}$ represents a rotation through the angle c about the z axis. Now the general rotation can be taken as a rotation c_3 about the z axis followed by a rotation c_2 about the y axis followed by a rotation c_1 about the z axis. In effect, the latter two rotations define the direction of the axis of rotation, and the first defines the angle of rotation about that axis. Since

$$\begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = U \underline{\rho}, \text{ where}$$

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (21)$$

and $U\tilde{U} = 1$ (22)
 ($\tilde{U}$ denotes the transpose of U), it is easily verified that these three rotations are represented by

$$T = \mathcal{J}(c_1) \tilde{U} \mathcal{J}(c_2) U \mathcal{J}(c_3). \quad (23)$$

To find the solutions of (15) we note that it must be valid for all $\theta_1, \theta_2, \theta_1', \theta_2'$ and all expansions K and the corresponding rotations specified by c_1, c_2, c_3 . Thus

$$K \frac{\partial r}{\partial \theta_i} = \frac{\partial r'}{\partial \theta_i} = \frac{\partial r'}{\partial \theta_j'} \frac{\partial \theta_j'}{\partial \theta_i} \quad (24)$$

with the understanding that summation is implied over any repeated suffix, such as j in (24). The matrix M and the vector $\underline{a}$ are defined by

$$M_{ij} = \frac{\partial \theta_j'}{\partial \theta_i} \quad (25)$$

$$a_i = \frac{\partial r}{\partial \theta_i}; \quad a_i' = \frac{\partial r'}{\partial \theta_i'} \quad (26)$$

Then (24) becomes

$$K \underline{a} = M \underline{a}' \quad (27)$$

Also from (15),

$$r \frac{\partial K}{\partial c_i} = \frac{\partial r'}{\partial c_i} = \frac{\partial r'}{\partial c_j'} \frac{\partial \theta_j'}{\partial c_i} \quad (28)$$

The matrix S and the vector β are defined by

$$S_{ij} = \frac{\partial \theta_j'}{\partial a_i} \quad (29)$$

$$\beta_i = \frac{\partial K}{\partial c_i} \quad (30)$$

Then (28) becomes

$$r \underline{\beta} = S \underline{a'} \quad (31)$$

Our aim now is to eliminate $\underline{a'}$ from (27) and (31) and thus obtain a set of differential equations for r in terms of θ_1 and θ_2 . This requires that M and S be expressed in terms of θ_1 and θ_2 , which is accomplished as follows: From (19)

$$\frac{\partial \rho_i'}{\partial \theta_i} = T \frac{\partial \rho}{\partial \theta_i} \text{ or } \frac{\partial \rho_m'}{\partial \theta_i} = T_{mj} \frac{\partial \rho_j}{\partial \theta_i} \quad (32)$$

But
$$\frac{\partial \rho_i'}{\partial \theta_i} = \frac{\partial \rho_i'}{\partial \theta_j'} \frac{\partial \theta_j'}{\partial \theta_i} \text{ or } \frac{\partial \rho_m'}{\partial \theta_i} = \frac{\partial \rho_m'}{\partial \theta_j'} \frac{\partial \theta_j'}{\partial \theta_i} \quad (33)$$

Define
$$A_{ij} = \frac{\partial \rho_j'}{\partial \theta_i'} ; A'_{ij} = \frac{\partial \rho_j'}{\partial \theta_i'} \quad (34)$$

Then (32) and (33) give

$$T \tilde{A} = \tilde{A'} \tilde{M} \text{ or } A \tilde{T} = M A' \quad (35)$$

Also
$$\frac{\partial \rho_m'}{\partial c_i} = \frac{\partial \rho_m'}{\partial \theta_j'} \frac{\partial \theta_j'}{\partial c_i} \quad (36)$$

and
$$\frac{\partial \rho_m'}{\partial c_i} = \frac{\partial T_{ml}}{\partial c_i} \rho_l \quad (37)$$

whence
$$S A' = N \quad (38)$$

where
$$N_{ij} = \frac{\partial T_{jl}}{\partial c_i} \rho_l \quad (39)$$

Note that (35) and (39) comprise 15 equations for the 16 elements of $M A'$ and S . In addition we have $A_{23}' = 0$ from (34) and (17), so that in principle we have enough equations to proceed.

At this point, we can simplify the analysis by choosing the coordinate system so that the axis of rotation is the z axis, i.e.,

$$c_1 = 0; c_2 = 0; c_3 = c \quad (40)$$

$$\text{and} \quad T = \mathcal{J}(c) \quad (41)$$

This does not involve any loss of generality provided $\frac{\partial}{\partial c_1}$ is carried out before (40). It is easily verified that (41) is equivalent to

$$\theta_1 = \theta_1'; \quad \theta_2 = \theta_2' + c \quad (42)$$

Hence from (25)

$$M = 1 \quad (43)$$

$\therefore$ (27) gives

$$K \underline{a} = \underline{a}' \quad (44)$$

Substitution in (31) gives

$$r \underline{\beta} = K S \underline{a} \quad (45)$$

which can be expressed more concisely by defining

$$\mathcal{R} = \ln r \quad \text{and} \quad k = \ln K \quad (46)$$

Then, from the definitions of $\underline{a}$ and $\underline{\beta}$ in (26) and (30), (45) can be expressed as

$$\frac{\partial k}{\partial c_i} = S_{ij} \frac{\partial \mathcal{R}}{\partial \theta_j} \quad (47)$$

To evaluate S , we can rearrange (38) as follows

$$S A' \tilde{A}' = N \tilde{A}'$$

(note that S is 3×2 , A is 2×3 and N is 3×3)

$$\therefore S = N \tilde{A}' (A' \tilde{A}')^{-1} \quad (48)$$

where $()^{-1}$ denotes the inverse matrix.

From (17) and (42)

$$\begin{aligned}\rho_1' &= \sin \theta_1 \cos(\theta_2 - c) \\ \rho_2' &= \sin \theta_1 \sin(\theta_2 - c) \\ \rho_3' &= \cos \theta_1.\end{aligned}\quad (49)$$

∴ from (34)

$$A' = \begin{pmatrix} \cos \theta_1 & \cos(\theta_2 - c) & \cos \theta_1 & \sin(\theta_2 - c) & -\sin \theta_1 \\ -\sin \theta_1 & \sin(\theta_2 - c) & \sin \theta_1 & \cos(\theta_2 - c) & 0 \end{pmatrix} \quad (50)$$

$$A' \tilde{A}' = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta_1 \end{pmatrix} \quad (51)$$

$$\therefore (A' \tilde{A}')^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{cosec}^2 \theta_1 \end{pmatrix} \quad (52)$$

$$\therefore \tilde{A}(A' \tilde{A}')^{-1} = \begin{pmatrix} \cos \theta_1 & \cos(\theta_2 - c) & -\frac{\sin(\theta_2 - c)}{\sin \theta_1} \\ \cos \theta_1 & \sin(\theta_2 - c) & \frac{\cos(\theta_2 - c)}{\sin \theta_1} \\ -\sin \theta_1 & 0 & 0 \end{pmatrix} \quad (53)$$

To evaluate N from (39) we use the general formula for T given in (23).

Thus

$$\begin{aligned}\frac{\partial T}{\partial c_1} &= J'(c_1) \quad \tilde{U} \quad J(c_2) \quad U J(c_3) \\ &= J'(0) \quad \tilde{U} \quad J(0) \quad U J(c), \text{ from (40)} \\ &= \begin{pmatrix} -\sin c & \cos c & 0 \\ -\cos c & -\sin c & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (54)$$

$$\begin{aligned}\frac{\partial T}{\partial c_2} &= J(0) \quad \tilde{U} \quad J'(0) \quad U J(c) \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ \cos c & \sin c & 0 \end{pmatrix} \end{aligned} \quad (55)$$

$$\frac{\partial T}{\partial c_3} = \nabla'(c) = \begin{pmatrix} -\sin c & \cos c & 0 \\ -\cos c & -\sin c & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (56)$$

Thus from (39)

$$N = \begin{pmatrix} \sin \theta_1 \sin(\theta_2 - c) & -\sin \theta_1 \cos(\theta_2 - c) & 0 \\ -\cos \theta_1 & 0 & \sin \theta_1 \cos(\theta_2 - c) \\ \sin \theta_1 \sin(\theta_2 - c) & -\sin \theta_1 \cos(\theta_2 - c) & 0 \end{pmatrix} \quad (57)$$

From (48), (53) and (57)

$$S = \begin{pmatrix} 0 & -1 \\ -\cos(\theta_2 - c) & \cot \theta_1 \sin(\theta_2 - c) \\ 0 & -1 \end{pmatrix} \quad (58)$$

From (47) and (58)

$$\frac{\partial k}{\partial c_1} = -\frac{\partial \mathcal{L}}{\partial \theta_2} \quad (59)$$

$$\frac{\partial k}{\partial c_2} = -\cos(\theta_2 - c) \frac{\partial \mathcal{L}}{\partial \theta_1} + \cot \theta_1 \sin(\theta_2 - c) \frac{\partial \mathcal{L}}{\partial \theta_2} \quad (60)$$

$$\frac{\partial k}{\partial c_3} = -\frac{\partial \mathcal{L}}{\partial \theta_2} \quad (61)$$

Now the left hand sides of (59), (60) and (61) are independent of $\theta_1 \theta_2$.

From (61)

$$\mathcal{L} = D \theta_2 + f(\theta_1) \quad (62)$$

where D is constant, and $f(\theta_1)$ any function of θ_1 . Substitution of (62) in (60) gives

$$\frac{df}{d\theta_1} \cos(\theta_2 - c) = \text{const} + D \cot \theta_1 \sin(\theta_2 - c) \quad (63)$$

which evidently has no non-trivial solution since both sides must match for all θ_1 and θ_2 . This implies that the rotation must be of a more

restricted kind than the general rotation represented by c_1, c_2, c_3 . If we consider a reduction to two degrees of freedom, we have

$$c_1 = \text{some function of } c_2, c_3 = g(c_2, c_3) \quad (64)$$

$$\text{and} \quad k = h(c_1, c_2, c_3) = i(c_2, c_3) \quad (65)$$

$$\therefore \frac{\partial k}{\partial c_2} = \frac{\partial i}{\partial c_2} = \frac{\partial h}{\partial c_1} \frac{\partial g}{\partial c_2} + \frac{\partial h}{\partial c_2} \quad (66)$$

Similarly, for $\frac{\partial k}{\partial c_3}$ and $\frac{\partial T}{\partial c_1}$. Thus in place of (59), (60) and (61) we have now two equations which, so far as θ_1 and θ_2 are concerned, are formed by taking linear combinations of corresponding sides of (59) with (60) and (61):

$$\frac{\partial k}{\partial c_2} = A \frac{\partial \mathcal{R}}{\partial \theta_2} - \cos(\theta_2 - c) \frac{\partial \mathcal{R}}{\partial \theta_1} + \cot \theta_1 \sin(\theta_2 - c) \frac{\partial \mathcal{R}}{\partial \theta_2} \quad (67)$$

$$\frac{\partial k}{\partial c_3} = B \frac{\partial \mathcal{R}}{\partial \theta_2} \quad (68)$$

Again there is no non-trivial solution. The rotation therefore has only one degree of freedom. Therefore T is a function of only one variable, which we shall call t . Now since $T \tilde{T} = 1$

$$|T| = \text{Determinant } T = 1 \quad (69)$$

$$\therefore \frac{d|T|}{dt} = 0. \quad (70)$$

But (70) is the condition that there be some ρ_0 and ρ_0' satisfying

$$\frac{d \rho_0'}{dt} = \frac{dT}{dt} \rho_0 = 0 \quad (71)$$

$\therefore \rho_0'$ is independent of t . The rotation therefore has a fixed axis represented by $\rho_0' = \rho_0$. Thus (59), (60) and (61) reduce to (61), giving the solution (62) where $f(\theta_1)$ is an arbitrary function. From (61) we also obtain

$$k = - Dc \quad (72)$$

Thus we conclude that the general formula for a surface which is transformed into itself by a rotation and an expansion is

$$\ln r = D\varphi + f(\theta) \quad (73)$$

where r, θ, φ are conventional spherical coordinates and the expansion factor K is related to the rotation angle c by

$$\ln K = k = - Dc \quad (74)$$

which follows directly from (61). Alternatively (73) can be expressed in the form

$$r = e^{D\varphi} F(\theta) . \quad (75)$$

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